SOME ANALOGUES OF EBERHARD'S THEOREM ON CONVEX POLYTOPES*

BY

B. GRÜNBAUM

ABSTRACT

Let p_k denote the number of k-gonal faces of a simple 3-polytope. Euler's relation leads to an equation between the p_k's which does not involve p_6. Eberhard proved in 1891 that every sequence of non-negative integers $(p_3, p_4, \ldots)$ satisfying this equation corresponds to a polytope for suitable values of p_6. In the present paper it is established that if $p_3 = p_4 = 0$ then every value $p_6 \geqq 8$ is suitable.

1. **Introduction.** Let the number of k-gonal faces of a convex polytope P in 3-dimensional Euclidean space E^3 be denoted by $p_k(P)$. Since Euler, it has been known that if P is a simple 3-polytope (that is, if each vertex of P is incident to only three edges), then

$$(*) \qquad \sum_{k \geqq 3} (6 - k)p_k(P) = 12.$$

In 1890, the blind geometer Victor Eberhard proved [3] the following converse of the above statement:

EBERHARD'S THEOREM. *If* $p_3, p_4, p_5, p_7, p_8, \cdots, p_n$ *are non-negative integers such that*

$$\sum_{k \geqq 3} (6 - k)p_k = 12,$$

then there exists a simple 3*-polytope* P *such that* $p_k(P) = p_k$ *for all* $k \geqq 3$, $k \neq 6$.

Neither Euler's equation (*), nor Eberhard's theorem, give any information concerning $p_6(P)$. It is well known (see, for example, [4], [3], [6], [5, Chapter 13]) that if the values of $p_k(P)$ for $6 \neq k \geqq 3$ are given, $p_6(P)$ may take infinitely many different values; in general, however, not all values of p_6 are possible for a given sequence $p_3, p_4, p_5, p_7, \cdots, p_n$. For example, if $p_3 = 3$, $p_4 = p_5 = 1$, $p_k = 0$ for $k \geqq 7$, then $p_6 \geqq 3$ [4, Theorem 26] and is odd [5, Theorem 13.4.1].

The known proofs of Eberhard's theorem yield polytopes P with values of $p_6(P)$ very large compared to, say, $\sum_{k \geq 7} p_k(P)$ (see [4], [5, Section 13.3]). An

* Research supported in part by the National Science Foundation under grant GP–7536
Received October 18, 1968

explanation for this may be found in a recent result of Barnette [1], which may be formulated as follows:

BARNETTE'S THEOREM. *Every simple 3-polytope P satisfies*

$$p_6(P) \geqq -\frac{1}{3}p_4(P) - \frac{2}{3}p_5(P) + \sum_{k\geqq 7}\left(\frac{k}{6}-2\right)p_k(P).$$

In particular, for simple 3-polytopes satisfying $p_4(P) = p_5(P) = 0$, the number of hexagons is necessarily rather large.

The aim of the present paper is to show that if only simple 3-polytopes with $p_3 = p_4 = 0$ are considered, then p_6 may be kept uniformly small and all except possibly few values of p_6 are possible. More precisely, we have

THEOREM 1. *Let $p_5, p_6, p_7, \cdots, p_n$ be non-negative integers such that*

$$(**) \qquad p_5 = 12 + \sum_{k\geqq 7}^{n} (k-6)p_k$$

and

$$p_6 \geqq 8.$$

Then there exists a simple 3-polytope P such that $p_k(P) = p_k$ for all $k \geqq 5$.

The main part of the paper is devoted to a proof of the weaker

THEOREM 2. *Let $p_5, p_7, p_8, \cdots, p_n$ be non-negative integers satisfying (**). Then there exists a simple 3-polytope P such that $p_k(P) = p_k$ for $k = 5, 7, 8, \cdots, n$, $p_3(P) = p_4(P) = 0$, and $p_6(P) \leqq 8$.*

We shall first prove Theorem 2, and then indicate how Theorem 1 may be established by performing minor modifications on the constructions used to prove Theorem 2. A number of additional results, and some open problems and conjectures are discussed in the last section.

Our task is greatly simplified by the following theorem of Steinitz (see [8], [9], [5, Section 13.1], [2]):

STEINITZ'S THEOREM. *If G is a 3-connected graph imbedded in the 2-sphere S^2, there exists a convex 3-polytope P such that the boundary complex of P is isomorphic to the cell complex defined on S^2 by G.*

This result enables us to prove Theorems 1 and 2 by constructing, for a given sequence $p_5, \cdots, p_n$ satisfying (**), a 3-valent, 3-connected planar graph G such that the numbers $p_k(G)$ of k-gonal "countries" defined by G satisfy $p_3(G) = p_4(G) = 0$, $p_k(G) = p_k$ for $k = 5, 7, 8, \cdots, n$, and $p_6(G) \leqq 8$ or else $p_6(G) = p_6$.

2. **Proof of Theorem 2.** For a given sequence $p_5, p_7, \cdots, p_n$ satisfying the assumption of Theorem 2 we shall construct, in a number of stages, a graph G as

described above. We shall first explain some of the main subconstructions, then see how they may be used to obtain G in general; in the end we shall deal with some exceptional cases, not covered by the main argument.

(1) *The "caterpillar"*. Let $n_1, \cdots, n_r$ be integers not smaller than 7, and let $r \geqq 4$. We construct the $(n_1, \cdots, n_r)$-*caterpillar* by taking an n_1-gon, $\cdots$, an n_r-gon, and by joining them, together with two hexagons and $6 + \sum_{i=1}^{r} (n_i - 6)$ pentagons, in the fashion indicated in Figure 1.

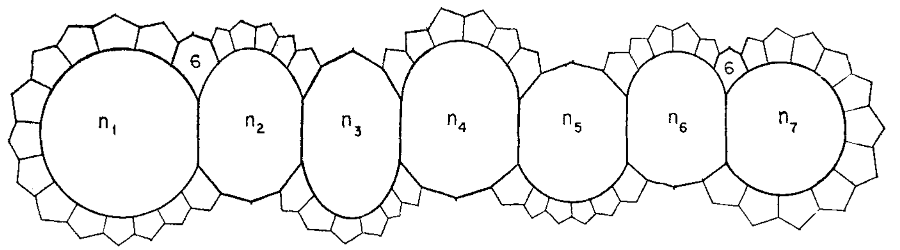

Fig. 1

One important feature of any caterpillar is its being a topological disc, such that the vertices on its boundary are, alternatingly, 2-valent and 3-valent. It is convenient to think of a caterpillar as a "toothed" disc the "valleys" on its rim corresponding to the 3-valent vertices. It is easy to see that each caterpillar is 3-connected between any two of its 3-valent vertices.

The caterpillars used in the main part of the proof will satisfy $n_1 \geqq 7$, $n_r \geqq 7$, $n_2 \geqq 12$, $n_{r-1} \geqq 12$, $n_i \geqq 11$ for $3 \leqq i \leqq r-2$. This will guarantee that each chain of contiguous pentagons along the rim of the caterpillar contains at least five pentagons.

(2) *The "welding"*. If we imbed a caterpillar in a 2-sphere, its complement is again a "toothed disc". (See Figure 2, in which a caterpillar with $r = 9$ is shown. By "pinching" that disc we may "weld together" almost all of the boundary of the caterpillar, as indicated in Figure 3. In the process we

(i) cover the whole sphere except for the two small topological discs shaded in Figure 3b;

(ii) change four of the caterpillar's pentagons into hexagons. This naturally implies that the caterpillar's polygons in the appropriate positions (separated by five valleys) had to be pentagons; thus it imposes a restriction on the freedom of preassigning the location of "pinching". However, since each chain of pentagons on the rim of the caterpillar contains at least five pentagons, it is easily checked that out of any five consecutive pinching positions at least one will have pentagons at the required places, and hence be possible.

(3) *The "plugs"*. Each of the two residual discs, remaining after the welding of the rim of the caterpillar to itself by (2), is an 11-gon with five 2-valent vertices.

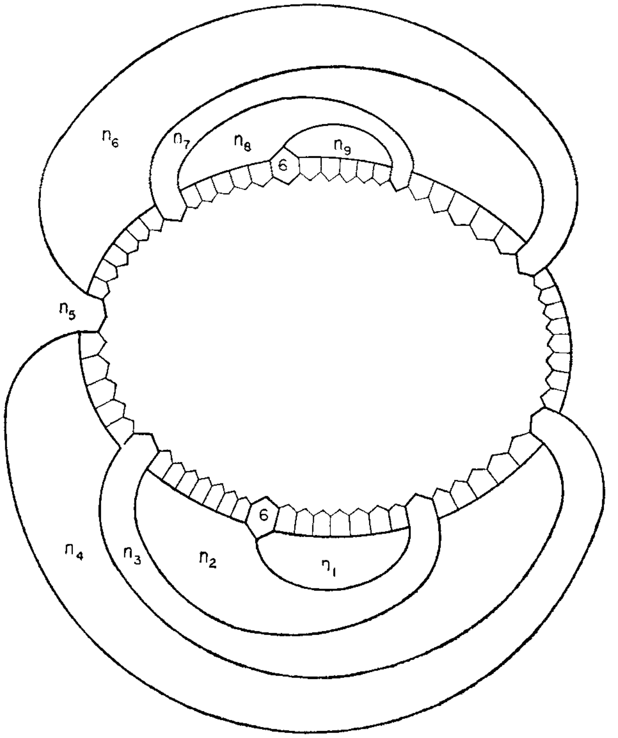

Fig. 2

It may be filled in by a "plug" consisting of 5 pentagons and one hexagon, as as shown in Figure 4.

(4) *Local insertions.* If a planar graph contains the configuration W of 4 pentagons shown in Figure 5a, it is possible to introduce two k-gons, $k \geqq 7$, into the graph without distrubing the graph outside the configuration. Moreover,

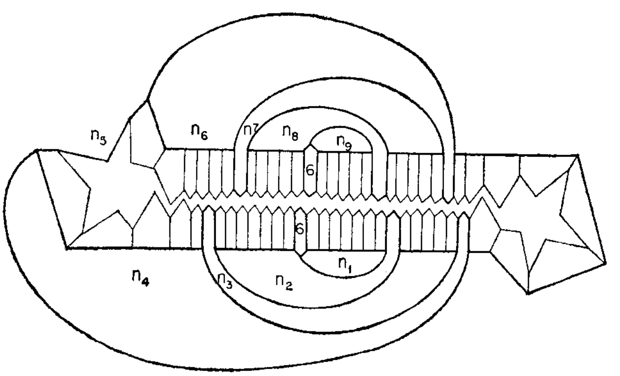

Fig. 3a

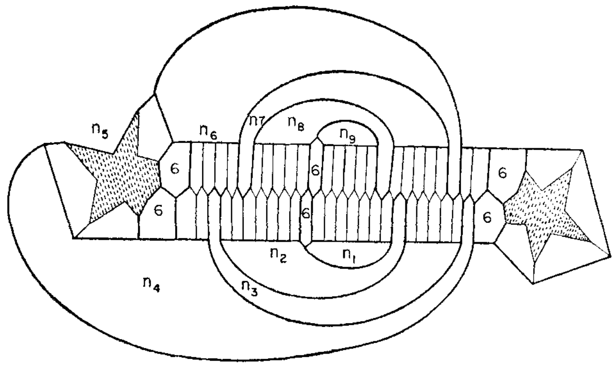

Fig. 3b

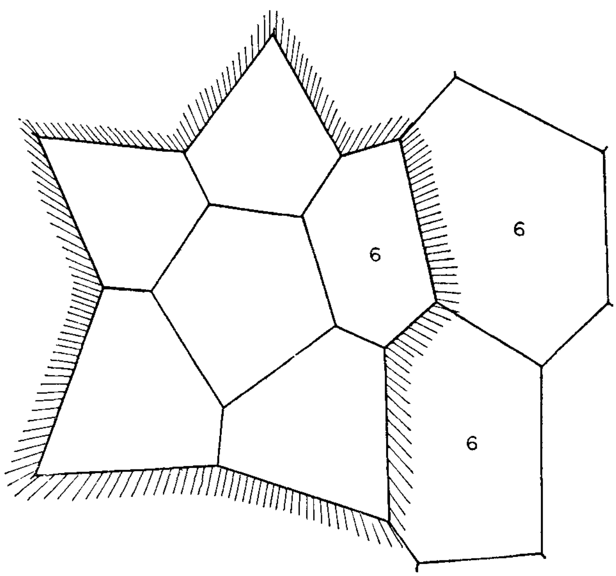

Fig. 4

this may be accomplished in such a fashion that the resulting configuration contains another copy of configuration W. The change is indicated (for $k = 9$) in Figure 5b, in which the new configuration W is shaded.

Since the configuration W is self-reproducing under such changes, we see that we may use it to insert an arbitrary *even* number of k_1-gons, an arbitrary even number of k_2-gons, etc. We shall use this construction only for $k = 7, 8, 9, 10, 11$.

Another change of a local character may be performed at each of the "plugs" introduced in (3). Indeed, considering one of them (Figure 4) together with the two hexagons adjacent to it (which arose from two pentagons in the rim of the caterpillar) we obtain the configuration of Figure 6, consisting of five pentagons

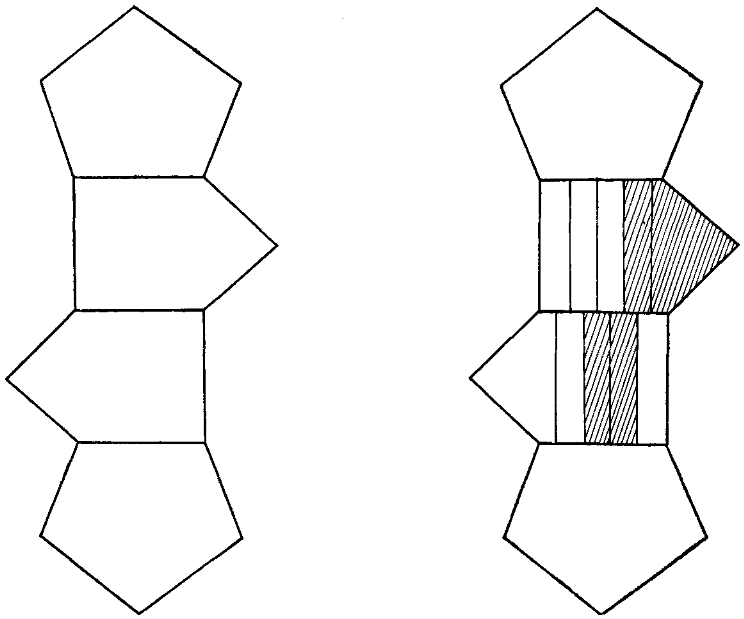

Fig. 5a, b

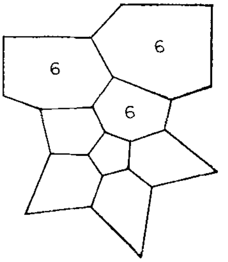

Fig. 6

and three hexagons. Modifying it in the ways indicated in Figure 7, we obtain configurations each of which consists, besides pentagons, of at most 3 hexagons and

either one heptagon,

or one octagon,

or one n-gon and one m-gon, where $7 \leqq n < m \leqq 11$ and $(n, m) \neq (7, 11)$.

Any of the procedures listed in (4) will be referred to as "local insertions".

* * *

Now we are ready to indicate the constructions needed to prove Theorem 2. Given the numbers $p_5, p_7, p_8, \cdots$, we will have the "general case" provided

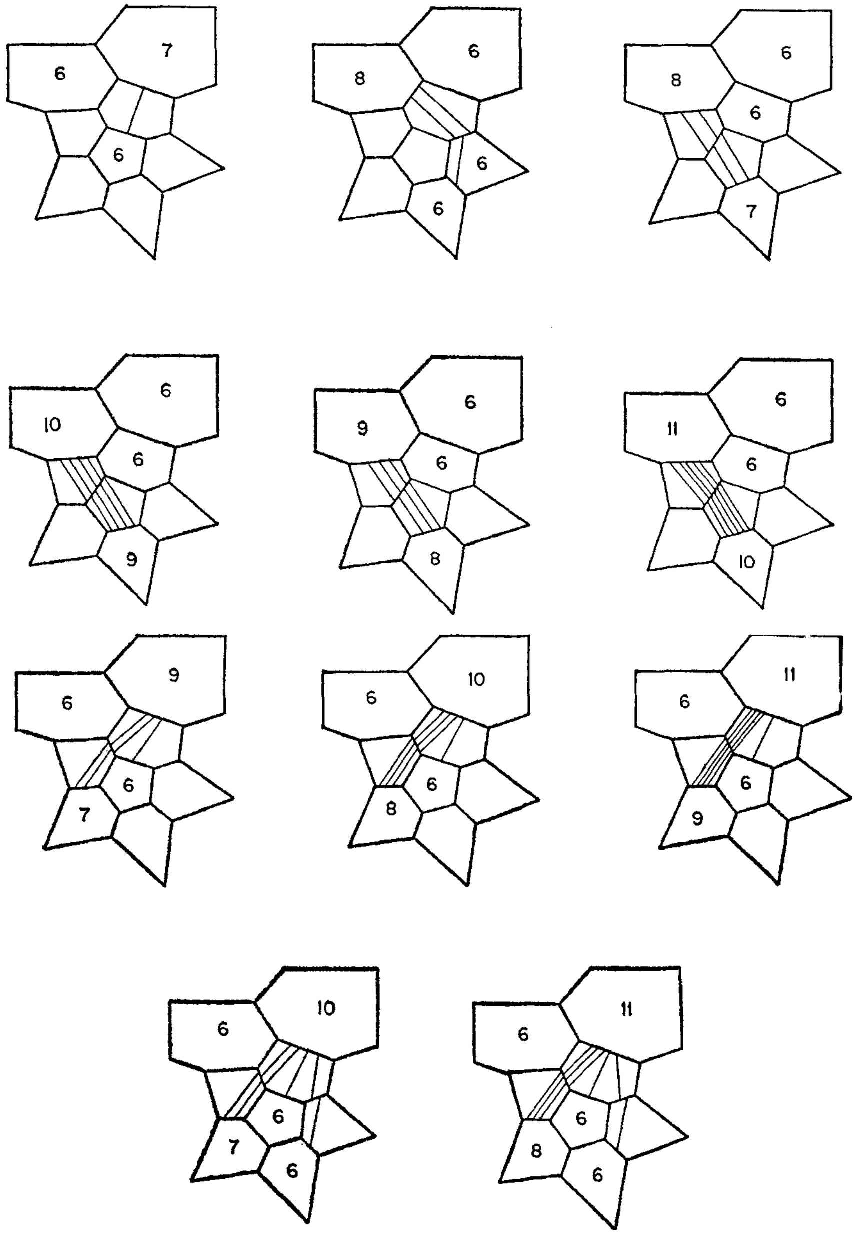

Fig. 7

either (i) $\sum_{k \geqq 12} p_k \geqq 4$;

or (ii) $\sum_{k \geqq 12} p_k = 3$ and $\sum_{k=7}^{11} p_k \geqq 1$:

or (iii) $\sum_{k \geqq 12} p_k = 2$, $p_9 + p_{10} + p_{11}$ is even and positive;

or (iv) $\sum_{k \geqq 12} p_k = 2$, $p_9 + p_{10} + p_{11}$ is odd, and $p_7 + p_8 \geqq 1$;

or (v) $\sum_{k \geqq 12} p_k = 2$, $p_9 + p_{10} + p_{11}$ is odd and $\geqq 3$, $p_{11} \geqq 1$, and $p_7 = p_8 = 0$.

An easy check reveals that in each subcase of the "general case" we may find polygons to form a caterpillar in which all chains of consecutive pentagons have length at least 5, all the polygons with at least 12 edges are in the caterpillar, as are (in some of the cases) some of the polygons with 7, 8, 9, 10, or 11 edges, while the remaining polygons with 7, 8, 9, 10, or 11 edges may be accomodated by local insertions. The only point that remains to be checked is the 3-connectedness of the resulting graph. However, if the "welding" of the caterpillar's rim is performed in such a fashion that at least one pair of pentagons adjacent to the two end-polygons of the caterpillar are brought together (as is shown in the example in Figure 3b), then the resulting graph is easily seen to be 3-connected, and the local insertions will not spoil its 3-connectedness. Since the two end-polygons of the caterpillar are at least heptagons, there are at least 10 different, consecutive ways of pinching which are satisfactory from the point of view of connectedness; as remarked earlier, we know that out of any five consecutive ways of "pinching" at least one is satisfactory from the point of view of having pentagons at the proper places. This completes the proof of Theorem 2 in the "general case".

* * *

We still have to construct the graphs in the following special cases:

(i) $\sum_{k \geqq 12} p_k = 2$, $p_9 + p_{10}$ is odd, and $p_7 = p_8 = p_{11} = 0$;

(ii) $\sum_{k \geqq 12} p_k = 2$ $p_{11} = 1$, $p_7 = p_8 = p_9 = p_{10} = 0$;

(iii) $\sum_{k \geqq 12} p_k = 2$, $p_9 = p_{10} = p_{11} = 0$;

(iv) $\sum_{k \geqq 12} p_k \leqq 1$.

In case (i), using the local insertions we see that it is enough to consider the case in which $p_9 + p_{10} = 1$. We shall in this case construct a "caterpillar" with only 3 polygons (having respectively $n_1 \geqq 12$, $n_2 = 9$ or 10, and $n_3 \geqq 12$) in its "body" (see Figures 8a and 8b). This caterpillar will be pinched in such a manner that either the pentagons a, a, or else the pentagons b, b, become adjacent and be transformed into hexagons. Due to the special position of the two hexagons present, at least one of those ways of pinching is permissible, and the resulting graph is 3-connected. This completes the proof in the special case (i).

The case (ii) is dealt with by a similar construction (see Figure 8c).

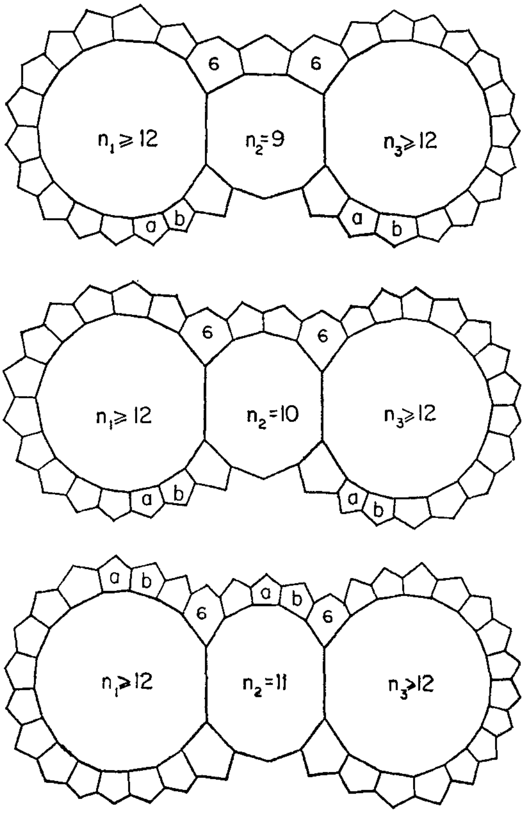

Fig. 8a, 8b, 8c.

In case (iii), let the two polygons with at least 12 sides have, respectively, n_1 and n_2 edges. We construct a caterpillar with just these two polygons (see Figure 9) and pinch it in such a fashion that either the pentagons a, a, or else the pentagons b, b, become adjacent and are transformed into hexagons. At least one of those two ways of pinching is permissible, and the resulting graph is 3-connected. The possibly still missing heptagons and octagons may now be taken care of by local insertions.

We note that the same procedure may be applied whenever $n_1 + n_2 \geqq 16$; hence the following subcases of case (iv) are also disposed of by this construction:

$$\sum_{k \geqq 12} p_k = 1 \text{ and } p_9 + p_{10} + p_{11} \text{ is odd;}$$

$$\sum_{k \geqq 12} p_k = 1,\ p_9 + p_{10} + p_{11} \text{ is even, and } p_7 + p_8 \geqq 1;$$

$$\sum_{k \geqq 12} p_k = 0,\ p_9 + p_{10} + p_{11} \text{ is even and positive:}$$

$$\sum_{k \geqq 12} p_k = 0,\ p_9 + p_{10} + p_{11} \text{ is odd, and } p_7 + p_8 \geqq 1;$$

$$\sum_{k \geqq 9} p_k = 0,\ p_8 \geqq 2;$$

Thus the only remaining cases are:

(a) $\sum_{k \geqq 12} p_k = 1$, $p_9 + p_{10} + p_{11}$ is even, and $p_7 = p_8 = 0$;

(b) $\sum_{k \geqq 12} p_k = 0$, $p_9 + p_{10} + p_{11} = 1, p_7 = p_8 = 0$;

(c) $\sum_{k \geqq 9} p_k = 0$, $p_8 = 1$;

(d) $\sum_{k \geqq 8} p_k = 0$, $p_7 \geqq 1$.

(Obviously, if $\sum_{k \geqq 7} p_k = 0$ then $p_5 = 12$ and G may be taken as the graph of the dodecahedron.)

For subcase (a) we note that (see Figure 10) the only n-gon present ($n \geqq 12$) may be surrounded by pentagons and serve as a caterpillar, the possibly present 9-, 10-, or 11- gons being accomodated by local insertions.

In subcase (b) we use one of the graphs in Figure 11.

For subcase (c) we take the graph of Figure 12a if p_7 is even, and the graph of Figure 12b if p_7 is odd; additional pairs of heptagons may be incorporated by local insertions.

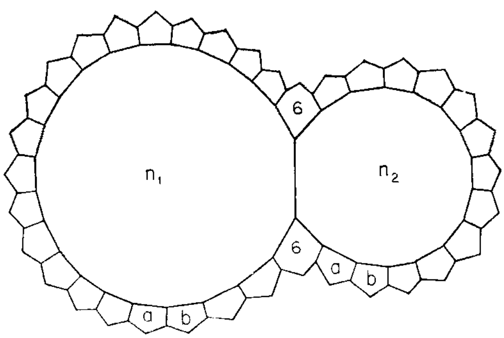

Fig. 9

Finally, in subcase (d) we take one of the graphs of Figure 13, and add pairs of heptagons as needed.

This completes the proof of the Theorem 2.

3. **Proof of Theorem 1.** We shall only sketch the proof of Theorem 1, leaving out the details of checking special cases. We first construct a graph G satisfying Theorem 2. The main step then consists in using the "local insertions" shown in Figures 14 and 15. The first adds one hexagon, while the second adds two or four hexagons, and replicates the starting configuration of six hexagons (shaded in Figure 15c). The few special cases in which the necessary starting configurations are absent must—and easily may—be treated separately.

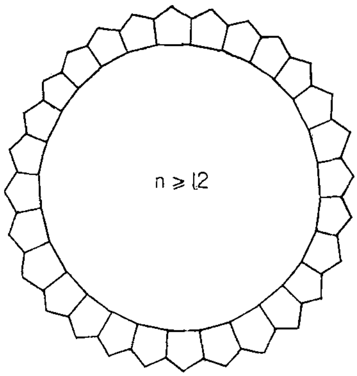

Fig. 10

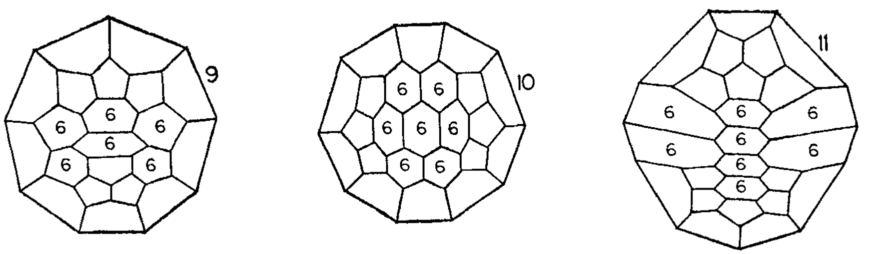

Fig. 11

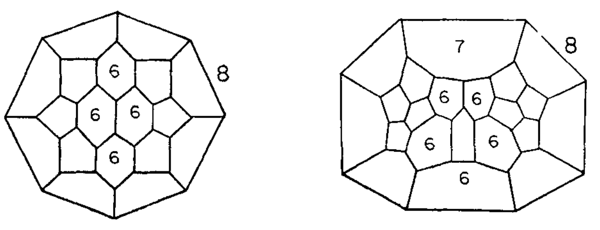

Fig. 12a, b

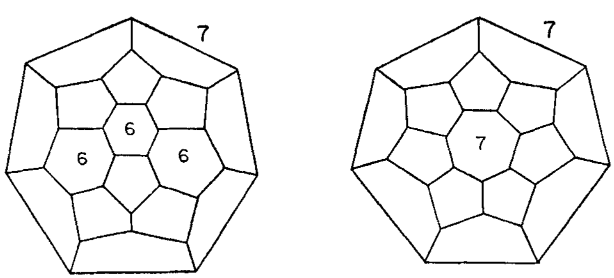

Fig. 13

This completes the sketch of the proof of Theorem 1.

4. **Remarks.** (i) The number 8 appearing in Theorems 1 and 2 is probably the best possible. However, the only case investigated in which even a determined effort

to find a polytope with less than 8 hexagons failed is that in which $p_{11} = 1$, $p_5 = 17$, and $p_k = 0$ for $k \neq 5, 6, 11$. Unfortunately, even in this case the need for at least 8 hexagons has not been definitively established.

ii) As indicated in the Introduction, Theorem 1 does not generalize to sequences in which p_3 and p_4 may be different from 0. However, as a generalization of Theorem 2 we venture the following

Conjecture 1. There exist constants a and b such that for any sequence $p_3, p_4, p_5, p_7, \cdots, p_n$ of non-negative integers satisfying Euler's relation (*) there exists a simple 3-polytope P such that $p_k(P) = p_k$ for $k \neq 6$, and

$$p_6(P) \leqq a(p_3 + p_4) + b.$$

The correct value of a is probably $a = 3$.

It is easy to show that if any estimate of the form indicated in the conjecture exists, it has to satisfy $a \geqq 1/2$.

There are some indications that a weakened form of Theorem 1 remains valid for more general sequences. We formulate it as

Conjecture 2. Given a sequence $p_3, p_4, p_5, p_7, \cdots, p_n$ of nonnegative integers satisfying Euler's relation (*), there exists a constant c such that either for each even, or else for each odd, p_6 with $p_6 \geqq c$, there exists a simple 3-polytope P satisfying $p_k(P) = p_k$ for all $k \geqq 3$.

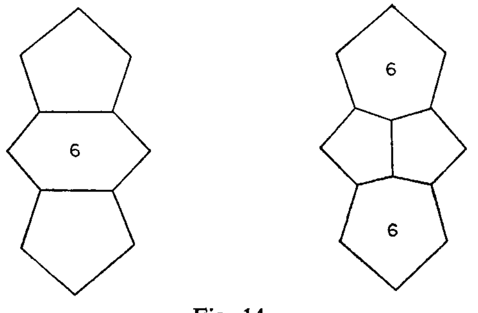

Fig. 14

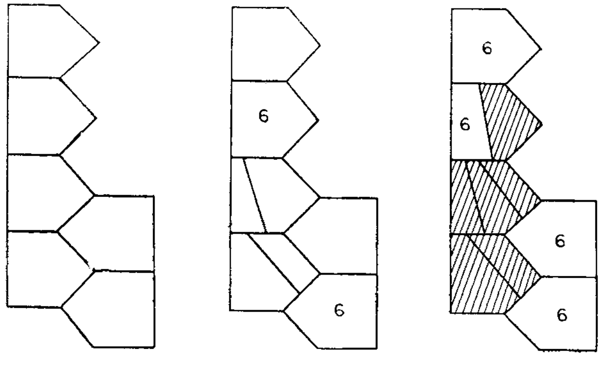

Fig. 15a, b, c

(iii) By a slight modification of the construction used in the proof of Theorem 2 we may establish

THEOREM 3. *Given non-negative integers* $p_5, p_6, p_7, \cdots, p_n$ satisfying $\sum_{k \geqq 5} (6-k)(p_k = 0$, *and* $p_6 \geqq 12$, *there exists a* 3-*valent*, 3-*connected graph* G *imbedded in the torus such that* $p_k(G) = p_k$ *for all* $k \geqq 5$.

The main change consists in omitting the insertion of the two "plugs" after welding the caterpillar's rim, replacing two of the caterpillar's pentagons by hexagons, and identifying the boundaries of the two holes. The changes and identifications are shown in Figure 17 for the caterpillar of Figures 2 and 3. Detailed consideration of the special cases leads to the constant 12 of Theorem 3. (We were not able to prove that 12 pentagons are indeed necessary for some sequence $p_5, \cdots, p_n$).

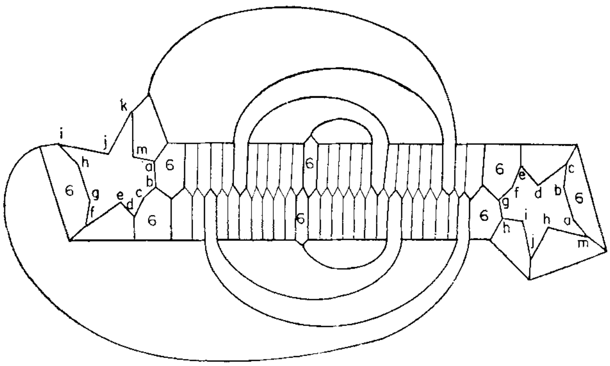

Fig. 17

Theorem 3 may probably be generalized in the following fashion:

Conjecture 3. If S is a surface of genus g, there exists a constant c_g such that for each sequence $p_5, p_6, p_7, \cdots, p_n$ of non-negative integers satisfying $\sum_{k \geqq 5}(6-k)p_k = 2(1-g)$ and $p_6 \geqq c_g$, there exists a 3-valent, 3-connected graph imbedded in S such that $p_k(G) = p_k$ for all $k \geqq 5$.

(iv) Using Theorem 13.2.5 of [5] instead of Steinitz's theorem, another easy modification of the method used here yields:

THEOREM 4. *Given non-negative even integers* $p_5, p_6, p_7, \cdots, p_n$ *satisfying equation* (**) *and* $p_6 \geqq 4$, *there exists a simple, centrally symmetric* 3-*polytope* P *such that* $p_k(P) = p_k$ *for all* $k \geqq 5$.

We also venture

Conjecture 4. There exist constants a and b such that given non-negative *even* integers $p_3, p_4, \cdots, p_n$ satisfying Euler's relation (*) and $p_6 \geqq a(p_3 + p_4) + b$, there exists a centrally symmetric simple 3-polytope P satisfying $p_k(P) = p_k$ for all $k \geqq 3$.

(v) As is well known there are 19 solutions with $p_i = 0$ for $i \geqq 7$ of Euler's equation (*). It was known already to Eberhard [4] that some of them require $p_6 > 0$ in order to correspond to a simple 3-polytope. Recently, efforts have made to determine, for each of the nineteen solutions, the precise set of values of p_6 possible in a simple 3-polytope. Table 1 gives the composite of those efforts. It is mainly based on results in [6], [5, Chapter 13], [7], and a private communication (December 1967) to the author by E. Jucovič (the entries marked (J) in Table 1 are due to Jucovič). Malkevitch [7] considered all the 19 cases and established in each case the admissibility of all the relevant values of p_6 which are

TABLE 1

	p_3	p_4	p_5	Excluded values of p_6	Remarks
1	4	0	0	2, all odd	
2	3	1	1	1, all even	
3	3	0	3	0,2,4	
4	2	3	0	1,3,7	
5	2	2	2		(J)
6	2	1	4	0	
7	2	0	6	1	(J)
8	1	4	1	0,1	
9	1	3	3		(J)
10	1	2	5	0	(J)
11	1	1	7	0,1	(J)
12	1	0	9	0,1,2,4	
13	0	6	0	1	
14	0	5	2	1	(J)
15	0	4	4		(J)
16	0	3	6		(J)
17	0	2	8		(J)
18	0	1	10	0,1	
19	0	0	12	1	

greater than some bound (such as 6 in case 3, 15 in case 4, etc.) The cases not covered by either Jucovič or Malkevitch were checked with the help of Brückner's [3] tables, or settled by ad hoc arguments. (*In case $p_3 = 2$, $p_4 = 3$, $p_6 = 15$, $p_k = 0$ for $k \neq 3,4,6$, an example was kindly communicated to the author by J. Malkevitch.)

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UNIVERSITY OF WASHINGTON,
SEATTLE

* Added on November 26, 1968.